Problem Sheet 7

Alex Kavvos

The following questions are about Modernised Algol.

1. Assuming that weakening is a rule of the system for both terms and commands, show that the following typing rules concerning various definable constructs are derivable.

   [Hint: Do them in the order presented below, so that you may assume that some rules are derivable while showing the next one.]

$$\frac{\Gamma \vdash_\Sigma m_1 \,\mathsf{ok} \qquad \Gamma, x : \mathsf{Nat} \vdash_\Sigma m_2 \,\mathsf{ok}}{\Gamma \vdash_\Sigma \{x \leftarrow m_1; m_2\} \,\mathsf{ok}} \qquad \frac{\Gamma \vdash_\Sigma m_1 \,\mathsf{ok} \qquad \Gamma \vdash_\Sigma m_2 \,\mathsf{ok}}{\Gamma \vdash_\Sigma \{m_1; m_2\} \,\mathsf{ok}}$$

$$\frac{\Gamma \vdash_\Sigma e : \mathsf{Cmd}}{\Gamma \vdash_\Sigma \mathsf{do}\, e \,\mathsf{ok}} \qquad \frac{\Gamma \vdash_\Sigma m \,\mathsf{ok} \qquad \Gamma \vdash_\Sigma m_1 \,\mathsf{ok} \qquad \Gamma \vdash_\Sigma m_2 \,\mathsf{ok}}{\Gamma \vdash_\Sigma \mathsf{if}\, m \,\mathsf{then}\, m_1 \,\mathsf{else}\, m_2 \,\mathsf{ok}}$$

$$\frac{\Gamma \vdash_\Sigma m \,\mathsf{ok} \qquad \Gamma \vdash_\Sigma m^\star \,\mathsf{ok}}{\Gamma \vdash_\Sigma \mathsf{while}\,(m)\{m^\star\} \,\mathsf{ok}} \qquad \frac{\Gamma, x : \tau \vdash_\Sigma m \,\mathsf{ok}}{\Gamma \vdash_\Sigma \mathsf{proc}\,(x : \tau)\,\{m\} : \tau \rightharpoonup \mathsf{Cmd}}$$

$$\frac{\Gamma \vdash_\Sigma e_1 : \tau \rightharpoonup \mathsf{Cmd} \qquad \Gamma \vdash_\Sigma e_2 : \tau}{\Gamma \vdash_\Sigma \mathsf{call}\, e_1(e_2) \,\mathsf{ok}}$$

2. Write down a transition sequence that begins with the following command-store pair, and ends in a final state, where $\mathsf{one} \stackrel{\text{def}}{=} \mathsf{succ}(\mathsf{zero})$ as usual. Moreover, show that the command is ok with $\Sigma \stackrel{\text{def}}{=} a$.

$$\{a := \mathsf{zero}; \mathsf{decl}\ b := \mathsf{one}\ \mathsf{in}\ \{x \leftarrow @\,b; \mathsf{ret}\ x\}\} \parallel \{a \mapsto \mathsf{one}\}$$

3. Complete the proofs of progress and preservation for Modernised Algol. As usual, do this in steps: first formulate a canonical forms lemma; then prove a substitution lemma; and then progress and preservation themselves.

   You are going to need the following 'extension' lemma.

   Lemma 1 (Extension)**.**

   - If $\Gamma \vdash_\Sigma e : \tau$ then $\Gamma \vdash_{\Sigma,\Sigma'} e : \tau$ for any appropriate Σ'.
   - If $\Gamma \vdash_\Sigma m \,\mathsf{ok}$ then $\Gamma \vdash_{\Sigma,\Sigma'} m \,\mathsf{ok}$ for any appropriate Σ'.

   The word 'appropriate' here means that the locations in Σ' do not clash with any of the locations in Σ. The Barendregt convention also means that any bound locations in e or m should be 'automatically' renamed to avoid clashing with Σ'.

   You are also going to need the following 'mobility' lemma.

   Lemma 2 (Mobility)**.** If $\vdash_\Sigma e : \mathsf{Nat}$ and $e\ \mathsf{val}$ then $\vdash_\emptyset e : \mathsf{Nat}$.

   This holds by repeated applications of canonical forms for Nat: if e is a value of natural number type it must be of the form $\mathsf{succ}^n(\mathsf{zero})$ for some $n \in \mathbb{N}$. Hence, starting with the typing rule Zero with $\Sigma = \emptyset$ and repeatedly applying Succ we can show that $\vdash_\emptyset e : \mathsf{Nat}$.

Finally, the substitution lemma you will need to prove (or assume!) is the following:

Claim 3 (Substitution)**.**

- If $\Gamma \vdash_\Sigma v : \sigma$, and $\Gamma, x : \sigma \vdash_\Sigma e : \tau$ then $\Gamma \vdash_\Sigma e[v/x] : \tau$.
- If $\Gamma \vdash_\Sigma v : \sigma$, and $\Gamma, x : \sigma \vdash_\Sigma m\,\mathsf{ok}$, then $\Gamma \vdash_\Sigma m[v/x]\,\mathsf{ok}$.

We may be using the letter v, but it need not be a value.

[For preservation, perform a simultaneous induction on $e \longmapsto e'$ and $m \parallel \mu \ \longmapsto_\Sigma m' \parallel \mu'$. Do a similar simultaneous induction on typing derivations for progress. You will need to use the canonical forms lemma in both, not just when proving progress.]