Problem Sheet 1

Alex Kavvos

1. Write down a derivation of the judgement

$$\mathsf{succ(succ(succ(zero)))\ odd}$$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{zero\ even}}\ \textsc{EvenZ}}{\mathsf{succ(zero)\ odd}}\ \textsc{Odd}}{\mathsf{succ(succ(zero))\ even}}\ \textsc{Even}}{\mathsf{succ(succ(succ(zero)))\ odd}}\ \textsc{Odd}$$

2. (i) Write down the rules that generate lists of natural numbers.
   (ii) Write down the associated induction principle.
   (iii) In your notation, write a derivation of the judgement that `[0, 1]` is a list.

Solution:

(i)

$$\textsc{Nil}\ \dfrac{}{\mathsf{nil\ list}} \qquad\qquad \textsc{Cons}\ \dfrac{n\ \mathsf{nat} \qquad xs\ \mathsf{list}}{\mathsf{cons}(n, xs)\ \mathsf{list}}$$

This is of course a mathematical restatement of the Haskell data types

```
data Nat = Zero | Succ Nat
data NatList = Nil | Cons Nat NatList
```

(ii) Let $\mathcal{P}$ be a property of the lists. If

- $\mathcal{P}(\mathsf{nil})$, and
- whenever n nat and $\mathcal{P}(xs)$ we know that $\mathcal{P}(\mathsf{cons}(n, xs))$

then $\mathcal{P}(xs)$ for all xs list.

(There is also another principle that additionally inducts on numbers.)

(iii)

$$\dfrac{\dfrac{}{\mathsf{zero\ nat}}\,\textsc{Zero} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{zero\ nat}}\,\textsc{Zero}}{\mathsf{succ(zero)\ nat}}\,\textsc{Succ} \qquad \dfrac{}{\mathsf{nil\ list}}\,\textsc{Nil}}{\mathsf{cons(succ(zero), nil)\ list}}\,\textsc{Cons}}{\mathsf{cons(zero, cons(succ(zero), nil))\ list}}\,\textsc{Cons}$$

3. Prove that the following rule is derivable.

$$\dfrac{n\ \mathsf{even}}{\mathsf{succ}(\mathsf{succ}(n))\ \mathsf{even}}$$

Solution: To prove that a rule is derivable we need to show that we can use a derivation of its premise as a 'module' or component in proving its conclusion.

Hence, let as assume we have a derivation of the premise:

$$\dfrac{\vdots}{n\ \mathsf{even}}$$

We can use this to derive the conclusion, using the following two rule applications:

$$\dfrac{\dfrac{\dfrac{\vdots}{n\ \mathsf{even}}}{\mathsf{succ}(n)\ \mathsf{odd}}\,\textsc{Odd}}{\mathsf{succ}(\mathsf{succ}(n))\ \mathsf{even}}\,\textsc{Even}$$

4. Prove that the following rule is admissible.

$$\dfrac{n\ \mathsf{even}}{n\ \mathsf{nat}}$$

(You might need to strengthen this statement a bit.)

Solution: In order to handle odd numbers, we strengthen the statement by proving that *both* of the following rules are admissible:

$$\dfrac{n\ \mathsf{even}}{n\ \mathsf{nat}} \qquad\qquad \dfrac{n\ \mathsf{odd}}{n\ \mathsf{nat}}$$

We do so by mutual induction on the derivations of n even and n odd.

Case(EvenZ). Suppose that n even holds by virtue of the rule ZeroE. This is to say that the derivation of n even is of the form

$$\frac{}{\textsf{zero even}}\ \textsc{EvenZ}$$

(and hence that $n = \textsf{zero}$). Then, by the rule Zero for natural numbers, we know that

$$\frac{}{\textsf{zero nat}}\ \textsc{Zero}$$

Recalling that $n = \textsf{zero}$, this proves that n nat.

Case(Even). Suppose that n even holds by virtue of the rule Even. That is to say that the derivation of n even is of the form

$$\frac{\dfrac{\vdots}{x\ \textsf{odd}}}{\textsf{succ}(x)\ \textsf{even}}\ \textsc{Even}$$

(and hence that $n = \textsf{succ}(x)$ for some x). Then, as x odd we have **by the induction hypothesis** that x nat. Given a derivation fo this judgement we can use the rule Succ of natural numbers to deduce that

$$\frac{\dfrac{\vdots}{x\ \textsf{nat}}}{\textsf{succ}(x)\ \textsf{nat}}\ \textsc{Succ}$$

Recalling that $n = \textsf{succ}(x)$, this proves that n nat.

Case(Odd). Suppose that n odd holds by virtue of the rule Odd. That is to say that the derivation of n odd is of the form

$$\frac{\dfrac{\vdots}{x\ \textsf{even}}}{\textsf{succ}(x)\ \textsf{odd}}\ \textsc{Odd}$$

(and hence that $n = \textsf{succ}(x)$ for some x). Then, as x even we have **by the induction hypothesis** that x nat. Given a derivation fo this judgement we can use the rule Succ of natural numbers to deduce that

$$\frac{\dfrac{\vdots}{x\ \textsf{nat}}}{\textsf{succ}(x)\ \textsf{nat}}\ \textsc{Succ}$$

Recalling that $n = \textsf{succ}(x)$, this proves that n nat.

Remark: Notice that the structure of the proof for the rule ODD is identical to that for EVEN; one could have just said "similar to the ODD case" and avoided a lot of work.

Remark: Notice that this proof is very similar to defining the following two Haskell functions by **mutual recursion**:

```
data Nat = Zero | Succ Nat

data Even = Zero | Succ Odd
data Odd = Succ Even

evens :: Even -> Nat
evens Zero = Zero
evens (Succ x) = Succ (odds x)

odds :: Odd -> Nat
odds (Succ x) = Succ (evens x)
```

5. (∗) All the judgements we have seen up to this point have been *unary*, in the sense that they referred to only one entity. For example, the judgement n nat only refers to the object n.

   However, judgements can have arbitrary *arity*, and can thus define arbitrary relations between an arbitrary number of objects. For example, the following *ternary* judgment $\mathsf{sum}(a, b, c)$ defines a relation between three objects: a, b and c.

$$\textsc{Base}\ \frac{b\ \mathsf{nat}}{\mathsf{sum}(\mathsf{zero}, b, b)} \qquad \textsc{Ind}\ \frac{\mathsf{sum}(a, b, c)}{\mathsf{sum}(\mathsf{succ}(a), b, \mathsf{succ}(c))}$$

   The judgement $\mathsf{sum}(a, b, c)$ can be written in more familiar notation as $a + b = c$.

   Such judgements can be used—amongst countless other things—to define functions. This exercise is about showing that the above rules define the addition function.

   (i) Write down a derivation of $\mathsf{sum}(\mathsf{succ}(\mathsf{zero}), \mathsf{succ}(\mathsf{zero}), \mathsf{succ}(\mathsf{succ}(\mathsf{zero})))$.

   (ii) Restate the above rules as a Haskell function on the data type

   ```
   data Nat = Zero | Succ Nat
   ```

   Does your code use pattern matching? Discuss its relation to the rules given above.

   (iii) Prove that if $\mathsf{sum}(a, b, c)$ then a nat, b nat, and c nat.

   (iv) (Existence) Prove that if a nat and b nat then there exists a c nat such that $\mathsf{sum}(a, b, c)$.

   (v) (Uniqueness) Prove that if $\mathsf{sum}(a, b, c)$ and $\mathsf{sum}(a, b, c')$ it must be that $c = c'$.

(vi) Conclude that $\mathsf{sum}(a, b, c)$ indeed defines a function on natural numbers.

Solution:

(ii)

```
sum :: Nat -> Nat -> Nat
sum Zero b = b
sum (Succ a) b = Succ (sum a b)
```

(iii) We prove the claim by induction on the derivation of $\mathsf{sum}(a, b, c)$.

Case(BASE). If the derivation is of the form

$$\dfrac{\dfrac{\vdots}{b\ \mathsf{nat}}}{\mathsf{sum}(\mathsf{zero}, b, b)}\ \textsc{Base}$$

then we know (i) that $\mathsf{zero}\ \mathsf{nat}$ by the rule ZERO, and (ii) that $b\ \mathsf{nat}$, because it is a premise of the rule BASE. So all three of zero, b, and b are natural numbers.

Case(IND). If the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\mathsf{sum}(a, b, c)}}{\mathsf{sum}(\mathsf{succ}(a), b, \mathsf{succ}(c))}\ \textsc{Ind}$$

then **by the inductive hypothesis (IH)** we know that $a\ \mathsf{nat}$, $b\ \mathsf{nat}$, and $c\ \mathsf{nat}$. Then,

- As $a\ \mathsf{nat}$, by the rule SUCC of natural numbers we deduce that $\mathsf{succ}(a)\ \mathsf{nat}$.
- As $c\ \mathsf{nat}$, by the rule SUCC of natural numbers we deduce that $\mathsf{succ}(c)\ \mathsf{nat}$.

Therefore, all three of $\mathsf{succ}(a)$, b and $\mathsf{succ}(c)$ are natural numbers.

(iv) We prove the claim by induction on the derivation of $a\ \mathsf{nat}$.

Case(ZERO). If the derivation is of the form

$$\dfrac{}{\mathsf{zero}\ \mathsf{nat}}$$

(so in fact $a \stackrel{\text{def}}{=} \mathsf{zero}$) then we can use the rule BASE to prove that

$$\dfrac{}{\mathsf{sum}(\mathsf{zero}, b, b)}$$

Hence, there *does* exist a c such that $\mathsf{sum}(a, b, c)$, and that c is in fact b.

Case(Succ). If the derivation is of the form

$$\dfrac{\dfrac{\vdots}{x\ \mathsf{nat}}}{\mathsf{succ}(x)\ \mathsf{nat}}\ \textsc{Succ}$$

(which is to say that $a = \mathsf{succ}(x)$ for some x). Then, by the **IH** applied to x nat, we know that there exists a c' such that $\mathsf{sum}(x, b, c')$. Given a derivation of this judgement, we use the rule Ind to deduce that

$$\dfrac{\dfrac{\vdots}{\mathsf{sum}(x, b, c')}}{\mathsf{sum}(\mathsf{succ}(x), b, \mathsf{succ}(c'))}\ \textsc{Ind}$$

Recalling that $a = \mathsf{succ}(x)$, we see that there *does* exist a c so that $\mathsf{sum}(a, b, c)$, namely $c \stackrel{\text{def}}{=} \mathsf{succ}(c')$.

Note: Notice that this proof is essentially just a regular natural number induction on a. Also, notice that it is very similar to the function definition given in Haskell above.

(v) We prove the claim by induction on the derivation of $\mathsf{sum}(a, b, c)$.

Case(Base). Suppose the derivation is of the form

$$\dfrac{}{\mathsf{sum}(\mathsf{zero}, b, b)}$$

(which is to say that $a = \mathsf{zero}$ and $c = b$). By assumption we also know that $\mathsf{sum}(\mathsf{zero}, b, c')$ for some c'. As the first component of this judgement is a zero, its derivation can only be of the form

$$\dfrac{}{\mathsf{sum}(\mathsf{zero}, b, b)}$$

(no other rule has a zero in the first component!). Thus, $c' = b = c$.

Case(Ind). Suppose the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\mathsf{sum}(x, b, y)}}{\mathsf{sum}(\mathsf{succ}(x), b, \mathsf{succ}(y))}\ \textsc{Ind}$$

(which is to say that $a = \mathsf{succ}(x)$ and $c = \mathsf{succ}(y)$ for some x and y). Consider the derivation of $\mathsf{sum}(a, b, c')$. As $a = \mathsf{succ}(x)$, this can only be of the form

$$\dfrac{\dfrac{\vdots}{\mathsf{sum}(x, b, y')}}{\mathsf{sum}(\mathsf{succ}(x), b, \mathsf{succ}(y'))}\ \textsc{Ind}$$

for some y' (no other rule matches the shape!). So, in fact, $c' = \mathsf{succ}(y')$.

At this point we have derivations of $\mathsf{sum}(x, b, y)$ and $\mathsf{sum}(x, b, y')$. By the **IH**, we obtain that $y = y'$. Hence $c = \mathsf{succ}(y) = \mathsf{succ}(y') = c'$.