~min sheet~

A little language of numbers + strings – Typing!

cont.

$$\text{VAR}\ \frac{}{\Gamma, x:\sigma \vdash x:\sigma}$$

$$\text{NUM}\ \frac{n \in \mathbb{N}}{\Gamma \vdash \text{num}[n] : \text{Num}}$$

$$\text{Str}\ \frac{s \in \Sigma^*}{\Gamma \vdash \text{str}[s] : \text{Str}}$$

$$\text{PLUS}\ \frac{\Gamma \vdash e_1 : \text{Num} \quad \Gamma \vdash e_2 : \text{Num}}{\Gamma \vdash \text{plus}(e_1; e_2) : \text{Num}}$$

$$\text{TIMES}\ \frac{\Gamma \vdash e_1 : \text{Num} \quad \Gamma \vdash e_2 : \text{Num}}{\Gamma \vdash \text{times}(e_1; e_2) : \text{Num}}$$

$$\text{CAT}\ \frac{\Gamma \vdash e_1 : \text{Str} \quad \Gamma \vdash e_2 : \text{Str}}{\Gamma \vdash \text{cat}(e_1; e_2) : \text{Str}}$$

$$\text{LEN}\ \frac{\Gamma \vdash e : \text{Str}}{\Gamma \vdash \text{len}(e) : \text{Num}}$$

$$\text{LET}\ \frac{\Gamma \vdash e_1 : \sigma_1 \quad \Gamma, x:\sigma_1 \vdash e_2 : \sigma_2}{\Gamma \vdash \text{let}(e_1; x.e_2) : \sigma_2}$$

bound

$$(\lambda x. \underbrace{\quad x \quad}_{e_2})\ e_1$$

$\text{let}(e_1;$ $\qquad 1 * 1$

$\text{let } x = \underbrace{1}_{e_1} \text{ in } \underbrace{x * x}_{e_2}$

Inversion

$$\text{PLUS}\quad \frac{\Gamma \vdash e_1 : \text{Num} \quad \Gamma \vdash e_2 : \text{Num}}{\Gamma \vdash \text{plus}(e_1; e_2) : \text{Num}}$$

inversion lemmata

Inversion: Suppose $\Gamma \vdash e : \tau$

1. (PLUS case) If $e = \text{plus}(e_1; e_2)$ then it must be that
   - $\tau = \text{Num}$
   - $\Gamma \vdash e_1 : \text{Num}$
   - $\Gamma \vdash e_2 : \text{Num}$

2. (CAT case) If $e = \text{cat}(e_1; e_2)$ then it must be that
   - $\tau = \text{String}$
   - $\Gamma \vdash e_1 : \text{Str}$
   - $\Gamma \vdash e_2 : \text{Str}$

Proven by
- Induction
- Inspection

Weakening

Weakening: If $\Gamma \vdash e : \tau$ and x is fresh then $\Gamma, x:\sigma \vdash e : \tau$

Intuition:

We have $x : \text{Num} \vdash \text{plus}(\text{num}[3]; \text{num}[2]) : \text{Num}$ and $z : \sigma$ is fresh

$$\text{PLUS}\ \frac{\text{Num}\ \dfrac{3 \in \mathbb{N}}{x : \text{Num} \vdash \text{num}[3] : \text{Num}} \qquad \dfrac{2 \in \mathbb{N}}{x : \text{Num} \vdash \text{num}[2] : \text{Num}}\ \text{N}}{x : \text{Num} \vdash \text{plus}(\text{num}[3]; \text{num}[2]) : \text{Num}}$$

$\Downarrow$

$$\text{PLUS}\ \frac{\text{Num}\ \dfrac{3 \in \mathbb{N}}{x : \text{Num}, z:\sigma \vdash \text{num}[3] : \text{Num}} \qquad \dfrac{2 \in \mathbb{N}}{x : \text{Num}, z:\sigma \vdash \text{num}[2] : \text{Num}}\ \text{N}}{x : \text{Num}, z:\sigma \vdash \text{plus}(\text{num}[3]; \text{num}[2]) : \text{Num}}$$

Substitution

meta-theoretic ops = ops that we will use in the specification of our language that are not constructs of the language we are defining

$$z[e/x] = \begin{cases} e & \text{if } z \equiv x \\ z & \text{if } z \not\equiv x \end{cases}$$

$$(num[n])[e/x] = num[n]$$
$$(str[s])[e/x] = str[s]$$

$$(plus(e_1;e_2))[e/x] = plus(e_1[e/x]; e_2[e/x])$$

Times, cat and len are analogous

$$(plus(num[1];x))[num[2]/x]$$
$$\to plus(num[1][num[2]/x]; x[num[2]/x])$$
$$\to plus(num[1]; x[num[2]/x])$$
$$\to plus(num[1]; num[2])$$

$$(plus(num[1];x))[num[2]/y]$$
$$\to plus(num[1]; x)$$

$$(let(e_1; y.e_2))[e/x] = let(e_1[e/x]; y.e_2[e/x])$$

$$let(y; \overset{z}{y}.len(\overset{z}{y}))[str[hi]/y]$$

not the same y!

→ referential clash

$$(let(x; \underbrace{\overset{z}{y}.x+\overset{z}{y}}))[y/x]$$

free

if we sub it in here, it will be bound

→ variable capture

Barendregt Convention

= we assume that everything has been alpha renamed already so this doesn't happen

Substitution: If $\Gamma \vdash e:\tau$ and $\Gamma, x:\tau \vdash u:\sigma$ then $\Gamma \vdash u[e/x]:\sigma$

Proof by induction on $\Gamma, x:\tau \vdash u:\sigma$

[Case: let]

GOAL = if $\Gamma \vdash e:\tau$ and $\Gamma, x:\tau \vdash let(e_1; y.e_2):\sigma$ then $\Gamma \vdash (let(e_1;y.e_2))[e/x]:\sigma$

P1 = $\Gamma \vdash e:\tau$
P2 = $\Gamma, x:\tau \vdash let(e_1;y.e_2):\sigma$
GOAL = $\Gamma \vdash (let(e_1;y.e_2))[e/x]:\sigma$

P2 must have had the following derivation (Inversion)

$$\frac{n \in N}{num}$$

SHAPE1 : SHAPE2

$$\dfrac{\dfrac{}{\Gamma, x:\tau \vdash e_1:\sigma_1} \quad \dfrac{}{\Gamma, x:\tau, y:\sigma_1 \vdash e_2:\sigma}}{\Gamma, x:\tau \vdash let(e_1;y.e_2):\sigma}\ LET$$

IH1 = if $\Gamma \vdash e:\tau$ and $\Gamma, x:\tau \vdash e_1:\sigma_1$ then $\Gamma \vdash e_1[e/x]:\sigma_1$

IH2 = if $\Gamma, y:\sigma_1 \vdash e:\tau$ and $\Gamma, y:\sigma_1, x:\tau \vdash e_1:\sigma_2$

then $\Gamma, y:\sigma_1 \vdash e_2[e/x]:\sigma_2$

... (see BP)